Check Digits and the ISBN
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When a book is published, it is assigned a number called the International Standard Book Number (ISBN). From 1970 until 2007, this number consisted of a 10-digit number, but it now consists of a 13-digit number. For example, the ISBN of *UMAP Modules (1984)* is 0-912843-07-1, but the ISBN of Dan Brown's *The lost Symbol* (2009) is 978-0-385-50422-5.

An International Standard Book Number consists of 4 parts (if it is a 10 digit ISBN) or 5 parts (for a 13 digit ISBN):
  (1) for a 13-digit ISBN, a prefix element - so far 978 or 979 are the only ones available
  (2) the registration group element, (language-sharing country group, individual country or territory)
  (3) the registrant element,
  (4) the publication element, and
  (5) a checksum character or check digit.

 In the 10-digit ISBN above, the first digit, 0, indicates that the book was published in an English-speaking country; the digits 912843 represent the publisher (COMAP, Inc.); 07 are the identifying numbers that COMAP has assigned to the book; the final digit, 1, is called the check digit of the ISBN.
To obtain the ISBN check digit:

I. Multiply the first nine digits of the ISBN by 10, 9, 8, 7, 6, 5, 4, 3, and 2, respectively, and then compute the sum of these nine products.

2. Find the remainder when this sum is divided by 11.

3. Subtract the remainder from 11 to determine the check digit. [NOTE: So that each possible check digit is a single digit, a check digit of 10 is written as X and a check digit is assigned the value of 0 if there is a remainder of 0.]

Formally, using modular arithmetic, we can say:
$$(10x_1 + 9x_2 + 8x_3 + 7x_4 + 6x_5 + 5x_6 + 4x_7 + 3x_8 + 2x_9 + x_{10}) \equiv 0 \pmod{11}.$$

For the ISBN 0-912843-07-1 that is discussed above:

I.   10(0) + 9(9) + 8(1) + 7(2) + 6(8) + 5(4) + 4(3) + 3(0) + 2(7) = 197

2.   197 = 11(17) + a remainder of 10.

3. Check digit = 11 – 10 = 1. (It checks!)

Exercise #1:
a. Check the check digit of the ISBN 91-825-6407-8.
b. Transpose any two adjacent digits of this ISBN and determine the check digit of the resulting number.

The 13-digit ISBN is calculated using a different formula. The ISBN-13 check digit, which is the last digit of the ISBN, must range from 0 to 9 and must be such that the sum of all the thirteen digits, each multiplied by its (integer) weight, alternating between 1 and 3, is a multiple of 10.

Formally, using modular arithmetic, we can say:

$$(x_1+3x_2+x_3+3x_4+x_5+3x_6+x_7+3x_8+x_9+3x_{10}+x_{11}+3x_{12}+x_{13}) \equiv 0 \pmod{10}.$$

The calculation of an ISBN-13 check digit begins with the first 12 digits of the thirteen-digit ISBN (thus excluding the check digit itself). Each digit, from left to right, is alternately multiplied by 1 or 3, then those products are summed modulo 10 to give a value ranging from 0 to 9. Subtracted from 10, that leaves a result from 1 to 10. A zero (0) replaces a ten (10), so, in all cases, a single check digit results.

For example, the ISBN-13 check digit of 978-0-306-40615-? is calculated as follows:

s = 9×1 + 7×3 + 8×1 + 0×3 + 3×1 + 0×3 + 6×1 + 4×3 + 0×1 + 6×3 + 1×1 + 5×3
 = 9 + 21 + 8 + 0 + 3 + 0 + 6 + 12 + 0 + 18 + 1 + 15
 = 93
93 / 10 = 9 remainder 3
10 – 3 = 7

Thus, the check digit is 7, and the complete sequence is ISBN 978-0-306-40615-7.

Formally, the ISBN-13 check digit calculation is:

$$x_{13} = (10 - (x_1 + 3x_2 + x_3 + 3x_4 + \cdots + x_{11} + 3x_{12}) \bmod 10) \bmod 10.$$

The ISBN saves much time when a book is ordered; it is much easier to use a standard format than to identify the book by its title, author, edition, or publisher. But the use of the ISBN does not mean that everything is going to be perfect; errors can occur when the ISBN is copied and sent incorrectly or becomes garbled when transmitted over a telephone line. The check digit can be used to detect such errors. From the experiences of people working with the ISBN, one of the most common errors is the transposition of adjacent digits.

Before we analyze this type of error, let's consider the ISBN 0-45-283527-8. Making the computations according to our rules, we see that the value of the check digit should be 5, not 8 as claimed. Of the first nine digits, which one (or ones) can be changed to obtain a check digit of 8? With some scrap paper and a pencil, we can try some possibilities; one of these is 0-45-283517-8. (Check it!)   Notice the check digit 5 had to be increased by 3 to obtain a check digit of 8, while the digit in the eighth position (which has a coefficient of 3 in the sum) is decreased from 2 to 1 so that the sum is decreased by 3(2 – 1). Hence, if the check digit is increased by 3, the sum has to be decreased by 3. This leads to the following result.

If the check digit of an ISBN is increased by x to obtain a new check digit, the sum of the original check digit has to be decreased by x.

From the above rules for determining the check digit, we have:

sum = 11 (natural number) + rem

and

check digit = 11 – rem,

so that

check digit = 11 – {sum -11(natural number)},

or,

check digit = 11(1 + natural number) – sum.

Adding x to both sides, we obtain

(check digit + x) = 11 (1 + natural number) – sum + x,

(new check digit) = 11 (1 + natural number) – (sum – x).

Hence, if a check digit is increased by x to obtain a new check digit, the sum is decreased by x. [Note: The result is true if "increased" is replaced with "decreased," and vice versa.]

We are now m a position to analyze the error when two adjacent digits are transposed. Let's consider the ISBN 0-45-283527-8 again. Although the correct check digit is 5, assume that 8 is correct and let's see what has to be done to transpose two adjacent digits in this ISBN to obtain another ISBN with the check digit 8. Since the check digit of 5 has to be increased by 3 to form the new check digit 8, the sum must be decreased by 3 according to the result above. Looking at the nine digits 0-45-283527 and their coefficients that form the sum, the pair of adjacent digits in the third and fourth positions from the left, 5 and 2, have coefficients 8 and 7, respectively, which contribute to the sum 8(5) + 7(2) = 54. Since the sum is to be decreased by 3 and the other terms of the sum will remain constant, then the 54 must be

decreased by 3, thus becoming 51, which is the same as 8(2) + 7(5). The key is that the transposed adjacent digits, the 5 and 2, differ by 3 also! Try this with the other pair of adjacent digits, 5 and 2.

If we transpose two adjacent digits in an ISBN that differ by z units, moving the large digit to the right, we increase the check digit by z units.

This means that to increase a check digit by z units, select two adjacent digits that differ by z units, where the larger of the two digits, a, is to the left of b. [Note: The result is also true when "increased" is replaced with "decreased" and the larger of the two adjacent digits is on the right.]

Let's summarize what has happened with ISBN 0-45-283527-8. Suppose that you work for a mail-order bookstore and you receive an order for 100 copies of a book. The order form contains only the above ISBN, along with the unit price. If we analyze the situation, it might mean that the order need not be returned to the sender for clarification. Assuming that the correct check digit is 8 and that the error is caused by transposing two adjacent digits, the check digit 5 has to be increased by 3, which in turn means that two adjacent digits have to differ by 3. Since this happens in two places, this leaves us with two possible corrections :

>   0-42-583527-8

or

>   0-45-283257-8.

The first corrected ISBN means that the book is by a different publisher than the one, "45," listed in the ISBN on the order form. If publisher "42" does not publish any books at the stated unit price or does not have any book identified as 583527, then the second corrected ISBN is a possibility. The second corrected ISBN means that another book, different from the one requested, is published by the same publisher. If that book does not sell for the stated unit price, then the first correctred ISBN is a possibility.

If the check digit is correct for the preceding nine digits, there is little chance that there is an error in an order. If the check digit is not correct for the preceding nine digits, an analysis, similar to the above one, can often resolve the problem.!

Exercise #2:
 COMAP, Inc. receives an order for 10 copies of *UMAP Modules, Tools for Teaching 1985*; the order includes the ISBN 0-918243-08-X. There is trouble. This book has an ISBN of 0-912843-08-X. How would you resolve the problem by transposing two adjacent digits, if you assume that the check digit is X?

Some Answers to the exercises:

1. a. Since 10(9) + 9(1) + 8(8) + 7(2) + 6(5) + 5(6) + 4(4) + 3(0) + 2(7) = 267 and 267 = 11(24) + 3,
        the check digit = 11 -3 = 8.
   b. Suppose that the 2 and 5 are transposed; the ISBN would have a check digit of 5.

2.
The ISBN 0-918243-08-X, the one that is sent and received, has a check digit of 4, which has to be increased by 6 to give a check digit of X. Since 8 -2 = 6, interchange the digits 8 and 2 to obtain the ISBN 0-912843-08-X, which is the ISBN for the given book. The error in the given ISBN was the transposition of two adjacent digits.