ACTIVITIES TO INTRODUCE MAXIMA-MINIMA PROBLEMS

By DAVID PLEACHER, John Handley High School, Winchester, VA 22601

Teacher's Guide

Introduction: Maxima and minima problems have interested mathematicians since the early Greeks. Heron is given credit for one of the most significant discoveries of his time—that when light travels from a point to a mirror and then to another point, it takes the shortest possible path. In daily life, practical problems involving maxima and minima arise frequently. Problems about the best shape, shortest distance, or maximum volume are often contemplated.

NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989) encourages the "exploration of calculus topics from both a graphical and numerical perspective," including the determination of maximum and minimum points of a graph. The activities presented here involve two standard problems from geometry and calculus—the volume of a box and the bank shot on a pool table.

Problem solving is emphasized as a method of inquiry and application. The students explore problems and describe results using graphical, numerical, and physical models. Students are encouraged to work in groups to discuss their ideas and make conjectures and convincing arguments.

Exploratory experiences are important for students before they encounter symbolic methods for solving the problems. The time spent in allowing them to experiment with these maxima-minima problems will be made up later in the students' understanding of the concepts because they will already know something about the problem before they must apply concepts and procedures from geometry and calculus.

Grade levels: 7–12. Seventh, eighth, and ninth graders can use sheets 1 and 2. Geometry students can also complete sheet 3, and calculus students can complete all four sheets.

Materials: Scissors, centimeter rulers, protractors, sheets of blank paper that can be cut to different sizes, graph paper, and the prepared worksheets. A computer is optional.

Objective: To explore maxima-minima problems using physical models and numerical, graphical, and symbolic methods

---

Edited by ***Mary Kim Prichard,*** *University of North Carolina at Charlotte, Charlotte, NC 28223*
Nadine Bezuk, *San Diego State University, San Diego, CA 92182*
Mally Moody, *Oxford High School, Oxford, AL 36203*

This section is designed to provide in reproducible formats mathematical activities appropriate for students in grades 7–12. This material may be reproduced by classroom teachers for use in their own classes. Readers who have developed successful classroom activities are encouraged to submit manuscripts, in a format similar to the "Activities" already published, to the editorial coordinator for review. Of particular interest are activities focusing on the Council's curriculum standards, its expanded concept of basic skills, problem solving and applications, and the uses of calculators and computers.

Directions: The activities require two class periods to complete. Junior high school teachers should use sheets 1 and 2 only. Have students try some of the extensions for homework. Teachers of geometry and second-year algebra can assign sheet 3 for homework after completing sheets 1 and 2. Calculus teachers can assign sheet 4 for homework also.

Sheet 1: Distribute sheet 1, scissors, a centimeter ruler, and a 20-cm-by-20-cm piece of paper to each student. Have students work in groups of four to collect data and fill in the chart. Make sure that students understand that the height of the box is the length of the sides of the cut-out squares. Allow students to examine boxes of several sizes. It is not obvious to many students that different volumes can be obtained! Challenge students to find the length of the side of the square that maximizes the volume of the box formed.

To assist students in this investigation, program 1 can be used. Students can use this program in a computer laboratory where they choose the range of values for the length of the square's side, or the program might be used in a whole-class demonstration. Students may want to run the program several times to zero in on the "best" length to use for their squares. It is important to discuss with students that these methods give only an approximate solution. Using the computer program, they can get fairly accurate approximations; other techniques, however, must be used to obtain exact solutions.

After students have collected their data, have them graph the height and volume on a coordinate system with the volume plotted on the vertical axis. Point out to students that the more points they plot, the better "picture" they will get of the graph. A smooth curve represents the infinitely many points that could be plotted. Calculus and second-year algebra students should be able to identify the equation of the function that is being graphed, $f(x) = x(20 - 2x)^2$. Calculus students should also set up the problem using algebra; solve it with derivatives; and compare results from their charts, graphs, and calculus solutions.

Extension: Distribute additional copies of sheet 1 so that students can find the size of the squares they must cut from sheets of paper of different sizes to maximize the volume of the box formed.

Sheet 2: Distribute a ruler, protractor, and a copy of sheet 2 to each student. Have each student locate a point on $\overline{AB}$ (call it E) and record the lengths of $\overline{CE}$ and $\overline{ED}$ and their sum in the table. After the students have had time to draw several paths and measure their distances, record some of their data on the chalkboard. Some students believe that the total distance from C to E to D remains constant regardless of the placement of point E. Challenge the students to find the point E that produces a minimum value for $CE + ED$. Make sure that students recognize that this point E is the only point on $\overline{AB}$ that makes angles CEA and DEB congruent.

Point out that another good application of this problem occurs in miniature golf. Tie together the activities on sheets 1 and 2 by pointing out that both activities seek a maximum or minimum value and that such values do exist. Examining the graphs should help make this fact clear. Calculus teachers can further tie together these activities by showing that the same method is used to solve both problems (i.e., by taking the first derivative, setting it equal to 0, and solving the resulting equation).

Sheet 3 (optional sheet for geometry, second-year algebra, and calculus classes): Distribute a copy of sheet 3 to each student. Have each student follow the directions and complete the proof.

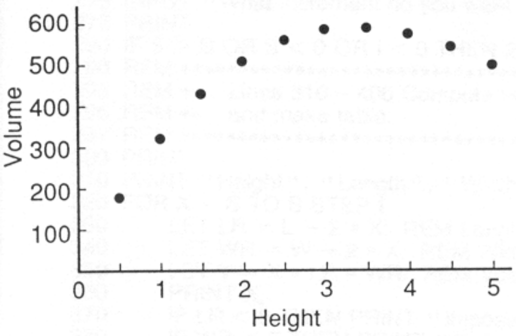
Sheet 4 (optional sheet for calculus classes): Distribute a copy of sheet 4 to each student. Have each student use the necessary algebra and calculus to complete the proof.

Answers

Sheet 1: 1. The following is a typical chart for a 20-cm-by-20-cm sheet of paper:

Height	Length	Width	Volume
0	20	20	0
0.5	19	19	180.5
1	18	18	324
1.5	17	17	433.5
2	16	16	512
2.5	15	15	562.5
3	14	14	588
3.5	13	13	591.5
4	12	12	576
4.5	11	11	544.5
5	10	10	500

2. The maximum volume, (16 000)/27, or ~592.6, is obtained by these dimensions: height, 10/3; length, 40/3; width, 40/3.
3. This graph represents the same set of data:

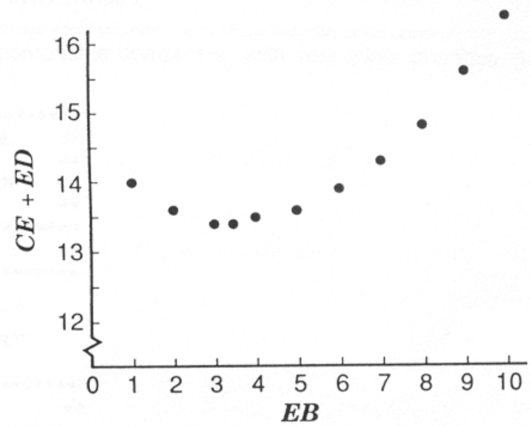


The height of the box with the greatest volume is between 3 and 3.5 cm. The exact value is 3 1/3 cm.

Sheet 2: 1. The following chart is typical:

EB	*CE*	*ED*	*CE* + *ED*
1	10.8	3.2	14
2	10	3.6	13.6
3	9.2	4.2	13.4
3.5	8.8	4.6	13.4
4	8.5	5	13.5

2. The measure of angles *CEA* and *DEB* is approximately 42 degrees.
3. The following graph represents all the data that would be found for $0 < EB < 10$:



Sheet 3: 3(*a*) SAS; 3(*b*) Corresponding parts of congruent triangles are congruent; 3(*c*) They are vertical angles; 3(*d*) transitive property; 4(*c*) Corresponding parts of congruent triangles are congruent; 4(*d*) addition property of equality; 4(*e*) Corresponding parts of congruent triangles are congruent; 4(*f*) addition property of equality; 4(*g*) triangle inequality

Sheet 4

1(*a*) $x^2 + 9$

1(*b*)

$$\frac{ds}{dx} = \frac{x-10}{\sqrt{x^2 - 20x + 136}} + \frac{x}{\sqrt{x^2+9}}$$

1(*c*) $x = 10/3$. One solution of the equation is $x = -10$, but it is not a sensible solution for the problem.

1(*d*) $s = \sqrt{181} \approx 13.5$ cm.

1(*e*) Measures are 10 cm and 9 cm; $CQ = \sqrt{181}$ cm.

2(*a*) $AE = c - x$.

2(*b*) $s = \sqrt{a^2 + (c-x)^2} + \sqrt{x^2 + b^2}$.

2(*c*)

$$\frac{ds}{dx} = \frac{x-c}{\sqrt{a^2 + (c-x)^2}} + \frac{x}{\sqrt{x^2+b^2}}$$

2(*d*)

$$x = \frac{b \cdot c}{b + a}$$

PROGRAM 1

This program was written in BASIC for IBM-compatible computers. With very minor changes, it can be adapted for use on any computer.

```
100 REM **************************************
110 REM **   This program will calculate the       **
120 REM **   volume of a box produced by           **
130 REM **   cutting congruent squares from        **
140 REM **   a rectangular sheet of paper.         **
150 REM **************************************
160 REM **   Use with Sheet 1.                     **
170 REM **************************************
180 CLS
200 PRINT " This program prints a table of"
210 PRINT "volumes for boxes made from a"
220 PRINT "given sheet of paper."
222 REM **************************************
225 REM **   Lines 230–270: Input data             **
226 REM **************************************
228 PRINT
230 INPUT "What is the length of your paper"; L
235 PRINT
240 INPUT "What is the width of your paper"; W
245 PRINT
250 INPUT "What is the smallest value for the length of the square that you want to use"; S
255 PRINT
260 INPUT "What is the largest value for the length of the square"; B
265 PRINT
270 INPUT "What increment do you want to use for changing the length of the square"; I
275 PRINT
280 IF S > B OR S < 0 OR I < 0 THEN 250
290 REM **************************************
295 REM **   Lines 310 – 400 Compute volumes **
296 REM **   and make table.                       **
297 REM **************************************
300 PRINT
310 PRINT "Height", "Length", "Width", "Volume"
320 FOR X = S TO B STEP I
330       LET LR = L – 2 * X:  REM Length of box
340       LET WR = W – 2 * X:  REM Width of box
350       LET V = X * LR * WR:  REM Volume of box
360       PRINT X,
370       IF LR < 0 THEN PRINT "Impossible", ELSE PRINT LR,
380       IF WR < 0 THEN PRINT "Impossible", ELSE PRINT WR,
390       IF V < 0 OR WR < 0 THEN PRINT "Impossible" ELSE PRINT V
400 NEXT X
410 END
```

BIBLIOGRAPHY

Byrne, Dan. "The Bank Shot." *Mathematics Teacher* 79 (September 1986):429–30, 487.

Garfunkel, J. "Exploring Geometric Maxima and Minima." *Mathematics Teacher* 62 (February 1969):85–90.

Grant, Nicholas. "Mathematics on a Pool Table." *Mathematics Teacher* 64 (March 1971):255–57.

Hamel, Thomas Ray, and Ernest Woodward. "Developing Mathematics on a Pool Table." *Mathematics Teacher* 70 (February 1977):154–63.

Immerzeel, George, and Melvin Thomas, comps. *Ideas from the Arithmetic Teacher, Grades 1–4, Primary.* Reston Va.: National Council of Teachers of Mathematics, 1982.

Kulm, Gerald. "Volume and Surface Area." *Mathematics Teacher* 68 (November 1975):583–86.

May, Beverly A. "Sharing Teaching Ideas: Reflections on Miniature Golf." *Mathematics Teacher* 78 (May 1985):351–53.

National Council of Teachers of Mathematics, Commission on Standards for School Mathematics. *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: The Council, 1989.

1. Take a 20-cm-by-20-cm piece of paper and cut out congruent squares from each corner. Fold up the sides to form a rectangular box. Determine the volume of the box and fill in the chart below.

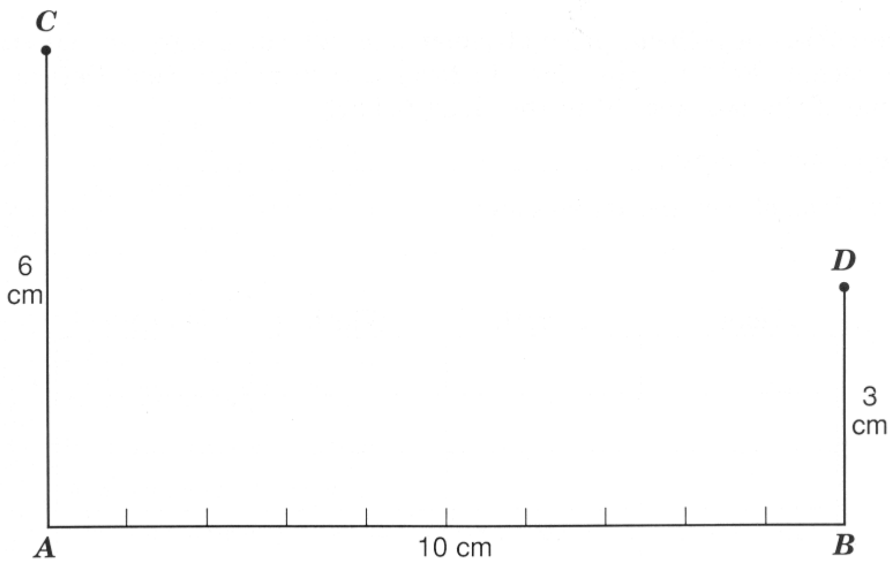
   Dimensions of paper: __________ × __________

   Dimensions of box and its volume:

Height	Length	Width	Volume

2. Collect more data by making additional models or using program 1. Add these data to the chart above and circle the dimensions that yield the maximum volume.

3. Plot the first and last columns of these data on a piece of graph paper, plotting the height on the horizontal axis and the volume on the vertical axis.



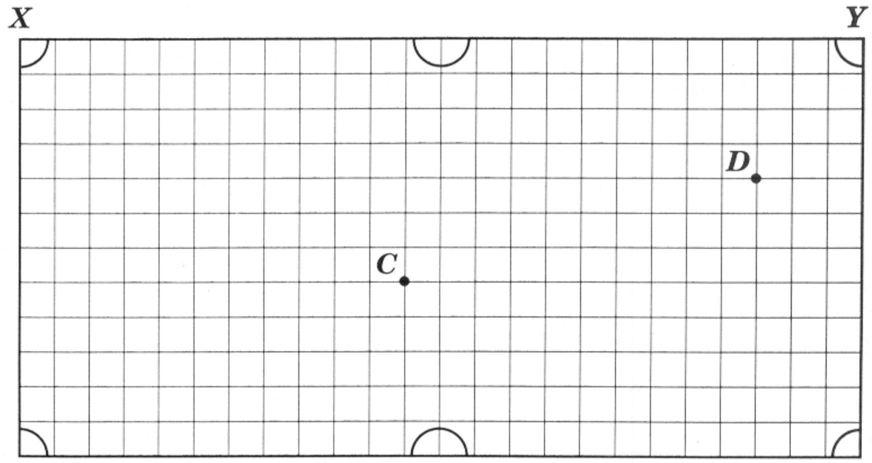
1. Choose a point on $\overline{AB}$ in the figure above, and label that point E. Measure CE and ED, and record the data in the chart below. Do the same for other points on $\overline{AB}$. Try to locate the point E on $\overline{AB}$ that results in the shortest distance from C to E to D.

EB	CE	ED	$CE + ED$

2. Measure angles CEA and DEB created by the point E that results in the shortest path. __________ and __________

3. Plot the data from the first and last columns of the chart on a piece of graph paper with EB on the horizontal axis and $CE + ED$ on the vertical axis.

1. In the diagram, mark the point on the cushion $\overline{XY}$ of the pool table to which you think the cue ball (labeled C) must be hit so that it rebounds and hits the other ball (labeled D).



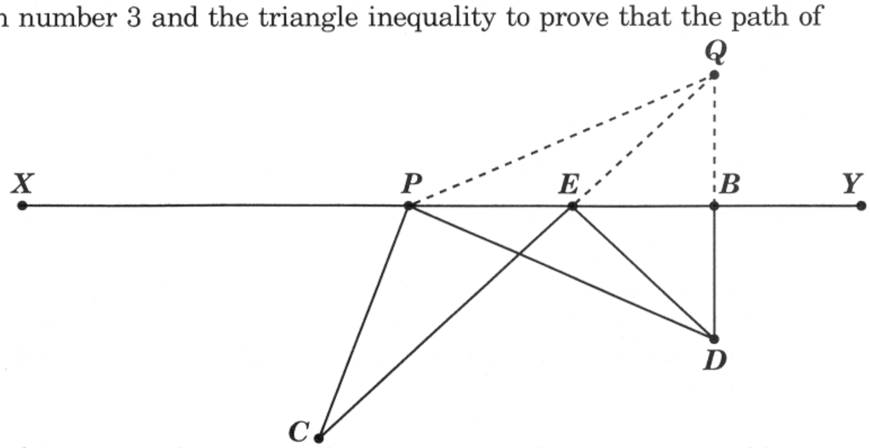
2. To locate this point geometrically, make the following constructions:

   (a) From point D, draw a line perpendicular to $\overline{XY}$; label the intersection B. Locate and label point Q on $\overleftrightarrow{DB}$ so that $BD = BQ$ (see the diagram in no. 4). (b) Draw $\overline{CQ}$. (c) The intersection of $\overline{CQ}$ and $\overline{XY}$ is the desired point; label that point E.

3. The path of the billiard ball is from C to E to D. Prove that the law of reflection holds true at point E, that is, show that $\angle CEP \cong \angle DEB$.

   (a) Why is $\triangle EQB \cong \triangle EDB$? ($b$) Why is $\angle QEB \cong \angle DEB$? ($c$) Why is $\angle CEP \cong \angle QEB$? ($d$) Why is $\angle CEP \cong \angle DEB$?

4. Use the proof in number 3 and the triangle inequality to prove that the path of the billiard ball is the minimum path from point C to $\overline{XY}$ and back to D.



   (a) In the diagram, choose a point on $\overline{XY}$ different from E. Label it P.

   (b) We want to prove that $CE + ED < CP + PD$ and therefore that this path is the shortest possible.

   (c) $ED = EQ$. Why? ______________________

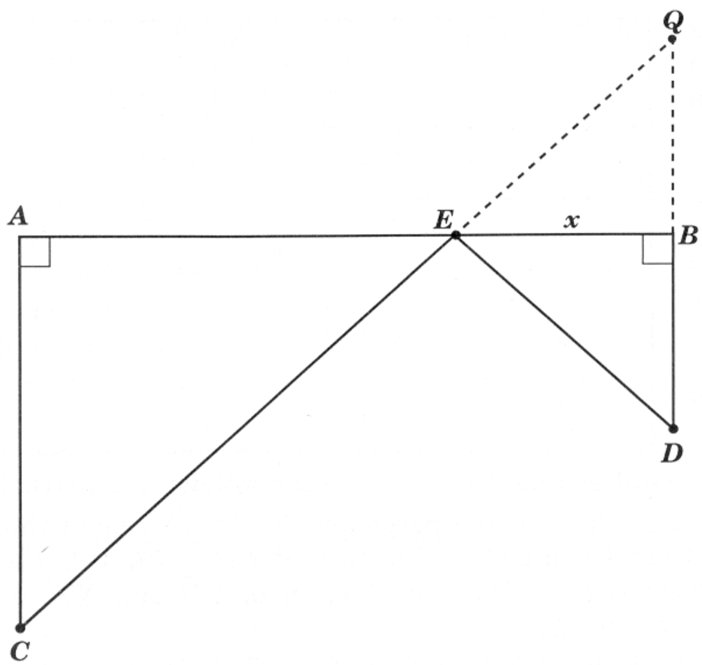
   (d) So $CE + ED = CE + EQ = CQ$. Why? ______________________

   (e) $DP = PQ$. Why? ______________________

   (f) So $CP + PD = CP + PQ$. Why? ______________________

   (g) Now $CP + PQ > CQ$. Why? ______________________

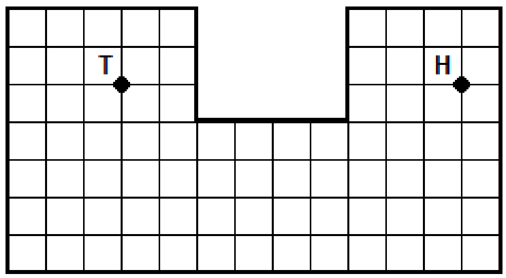
   (h) By substituting in (g), the desired result is obtained: $CP + PD > CE + ED$, or $CE + ED < CP + PD$.

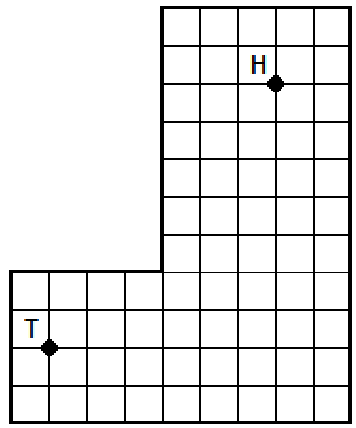


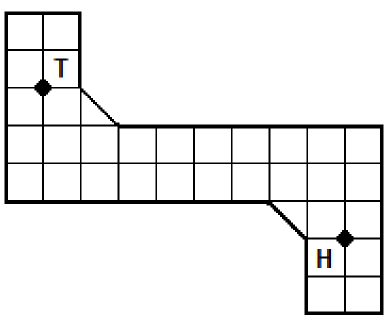
1. To determine the point E on the $\overline{AB}$ that yields the minimum distance from C to E to D in the figure, let x represent the distance from B to E. If $AC = 6$, $AB = 10$, and $BD = 3$, then the following equation for the total distance s can be written:
   - (a) $s = \sqrt{36 + (10\ x)^2} + \sqrt{\_\_\_\_\_\_ + \_\_\_\_\_\_}$.
   - (b) Simplify s and find ds/dx: ______________________________
   - (c) Set ds/dx equal to 0 and simplify, then solve for x: $x =$ ______________
   - (d) Substitute the value for x into the equation in step a, and find the value of s: $s =$ __________.
   - (e) Check this value of s by using the Pythagorean theorem to find CQ. Extend $\overline{QD}$ through D to form a right triangle with $\overline{CQ}$ as its hypotenuse. The two legs have lengths ________ and ________. Then $CQ =$ __________.

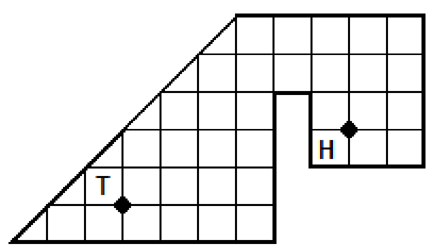
2. Solve the more general problem where $AC = a$, $AB = c$, and $BD = b$. Let $x = BE$.
   - (a) Then $AE =$ __________.
   - (b) $s = \sqrt{\_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_} + \sqrt{\_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_}$.
   - (c) Then find ds/dx: ______________________________
   - (d) Set ds/dx equal to 0 and solve for x: $x =$ __________.

In 1927, Garnet Carter built the first miniature golf course on Lookout Mountain in Tennessee. In each diagram below, show how a player would hit the ball from T so that it would go into the hole at H. Use the information that you learned in the pool table geometry above to determine the path.



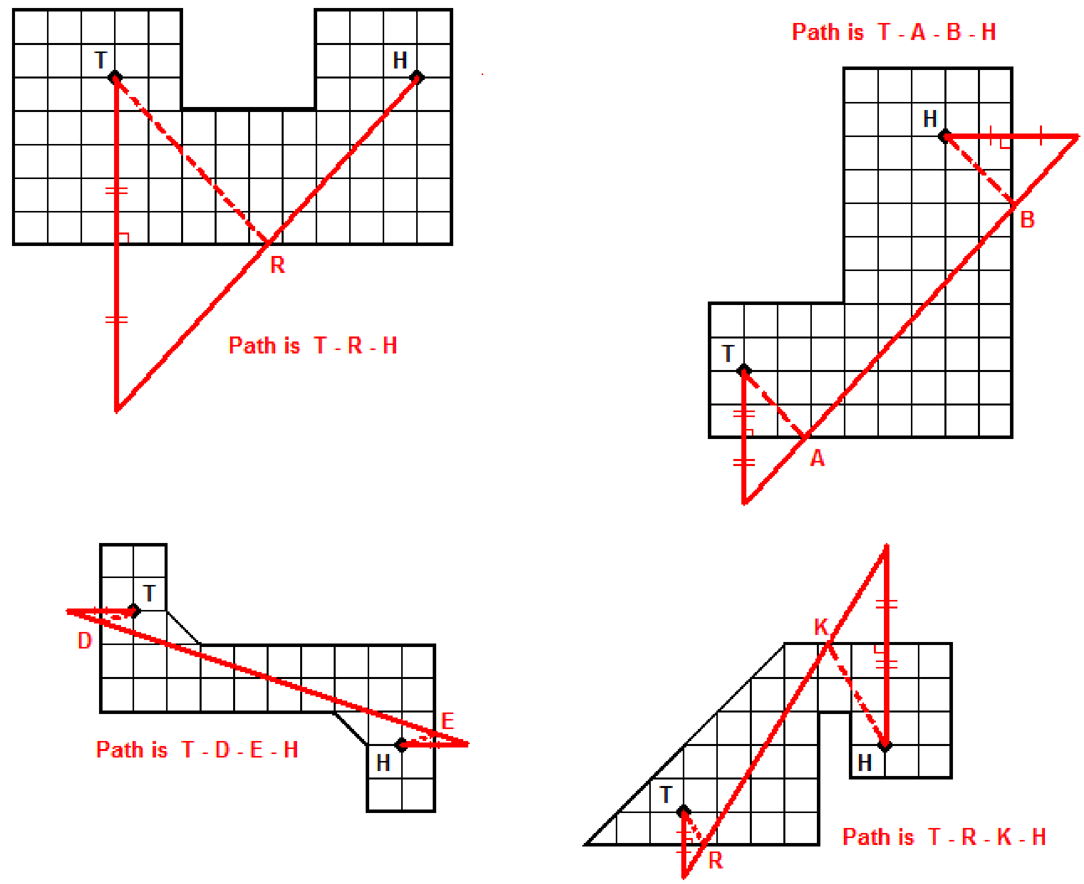






Click here for a tough golf shot!

Answer to Miniature Golf Holes:



Some Possible Paths