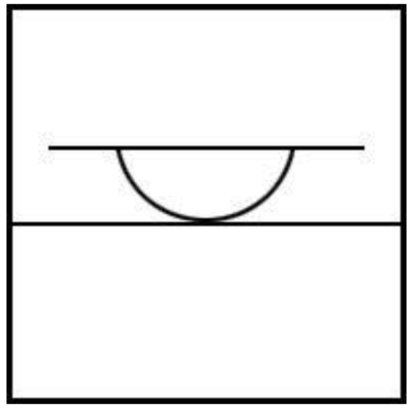
A Droodle for A.P. Calculus Exam

Puzzle and Answer Key by David Pleacher



"A Droodle is a borkley looking sort of drawing that doesn't make any sense until you know the correct title." – Roger Price

Caption for the picture:

F	R	I	E	D		E	G	G	,	S	U	N	N	Y		S	I	D	E		D	O	W	N	.
15	14	5	13	8		13	11	11		9	6	4	4	3		9	5	8	13		8	16	10	4	

To determine the title to this droodle, which was created by Roger Price and published in his book called *Droodles*, solve the 16 A.P. Calculus problems (from the 1993 AB Exam).
Then find the answers to each problem from the choices below.
Replace each numbered blank with the letter corresponding to the answer for that problem.
A calculator should not be used on the following problems except for #7.

P 1. $\displaystyle\int_0^{\frac{1}{2}} \frac{8\,dt}{1+4t^2} =$

Let $u = 2t$

Then $du = 2\,dt$

$$\int \frac{8\,dt}{1+4t^2} = \int \frac{4\,du}{1+u^2} = 4Tan^{-1}(u) = 4Tan^{-1}(2t)$$

$$\int_0^{\frac{1}{2}} \frac{8\,dt}{1+4t^2} = 4Tan^{-1}(2t)\Big|_0^{\frac{1}{2}} = 4Tan^{-1}\left(2\left(\frac{1}{2}\right)\right) - 4Tan^{-1}(0) = \pi - 0 = \pi$$

_J_ 2. At what value of x does the graph of $y=\frac{1}{x^2}-\frac{1}{x^3}$ have a point of inflection?

First, determine the second derivative, which gives us concavity. Set it equal to zero.

$$y'=-2x^{-3}+3x^{-4}$$

$$y''=6x^{-4}-12x^{-5}$$

$$0=6x^{-5}(x-2)$$

$$x=2$$

So, x = 2 is a possible point of inflection. Check concavity on either side to make sure that it changed. At x = 1, f''(x) is negative; at x = 10, f''(x) is positive.

_Y_ 3. An antiderivative for $\frac{1}{x^2-2x+2}$ is

$$\int\frac{dx}{x^2-2x+2}=\int\frac{dx}{\left(x^2-2x+1\right)+2-1}=\int\frac{dx}{\left(x-1\right)^2+1}$$

$$=\int\frac{dx}{1+\left(x-1\right)^2}=Tan^{-1}\left(x-1\right)+K$$

_N_ 4. How many critical points does the function $f\left(x\right)=\left(x+2\right)^5\left(x-3\right)^4$ have? Three.

Critical points are when the first derivative is zero.

$$f\left(x\right)=\left(x+2\right)^5\left(x-3\right)^4$$

$$f'\left(x\right)=\left(x+2\right)^5\bullet\frac{d}{dx}\left(x-3\right)^4+\left(x-3\right)^4\bullet\frac{d}{dx}\left(x+2\right)^5$$

$$f'\left(x\right)=\left(x+2\right)^5\left(4\left(x-3\right)^3\right)+\left(x-3\right)^4\left(5\left(x+2\right)^4\right)$$

$$=\left(x-3\right)^3\left(x+2\right)^4\left(4\left(x+2\right)+5\left(x-3\right)\right)$$

$$=\left(x-3\right)^3\left(x+2\right)^4\left(4x+8+5x-15\right)$$

$$=\left(x-3\right)^3\left(x+2\right)^4\left(9x-7\right)$$

$$x=3,-2,\frac{7}{9}$$

_I_ 5. If $f(x)=\left(x^2-2x-1\right)^{\frac{2}{3}}$, then $f'(0)$ is

$$f'(x)=\frac{2}{3}\left(x^2-2x-1\right)^{\frac{-1}{3}}\bullet\frac{d}{dx}\left(x^2-2x-1\right)$$

$$f'(x)=\frac{2}{3}\left(x^2-2x-1\right)^{\frac{-1}{3}}\bullet(2x-2)$$

$$f'(0)=\frac{2}{3}\left(0^2-2(0)-1\right)^{\frac{-1}{3}}\bullet\left(2(0)-2\right)$$

$$f'(0)=\frac{2}{3}\bullet\frac{1}{\sqrt[3]{-1}}\bullet(-2)=\frac{-4}{-3}=\frac{4}{3}$$

_U_ 6. $\frac{d}{dx}\left(2^x\right)=2^x\ln 2$

Let $y=\left(2^x\right)$

Then $\ln(y)=\ln\left(2^x\right)$

$\ln(y)=x\ln(2)$

Then $\frac{d}{dx}(\ln y)=\frac{d}{dx}(x\ln(2))$

$$\frac{1}{y}\frac{dy}{dx}=\ln(2)$$

$$\frac{dy}{dx}=y\ln(2)=\left(2^x\right)\ln 2$$

_K_ 7. A particle moves along a line so that at time t, where $0\le t\le\pi$, its position is given by $s(t)=-4\cos t-\frac{t^2}{2}+10$. What is the velocity of the particle when its acceleration is zero?

$v(t)=4\sin t-t$

$a(t)=4\cos t-1$

$0=4\cos t-1$

$\cos t=\frac{1}{4}$

$t=1.318$

$v(1.318)=4\sin(1.318)-(1.318)=2.55$

_D_ 8. $\lim_{\theta\to 0}\left(\frac{1-\cos\theta}{2\sin^2\theta}\right)=$

$$\lim_{\theta\to 0}\left(\frac{1-\cos\theta}{2\sin^2\theta}\right)=\lim_{\theta\to 0}\left(\frac{1-\cos\theta}{2\sin^2\theta}\right)\bullet\left(\frac{1+\cos\theta}{1+\cos\theta}\right)=$$

$$=\lim_{\theta\to 0}\left(\frac{1-\cos^2\theta}{(2\sin^2\theta)(1+\cos\theta)}\right)=\lim_{\theta\to 0}\left(\frac{\sin^2\theta}{(2\sin^2\theta)(1+\cos\theta)}\right)$$

$$=\lim_{\theta\to 0}\left(\frac{1}{(2)(1+\cos\theta)}\right)=\left(\frac{1}{(2)(1+\cos 0)}\right)=\frac{1}{4}$$

_S_ 9. The region enclosed by the x-axis, the line $x=3$, and the curve $y=\sqrt{x}$ is rotated about the x-axis. What is the volume of the solid generated?
First sketch the diagram. Then note that you can use the disk method to solve for the volume. $V=\pi r^2 h$

$$V=\int_0^3 \pi\left(\sqrt{x}\right)^2 dx=\pi\int_0^3 x\,dx=\pi\left.\frac{x^2}{2}\right|_0^3=\frac{9\pi}{2}$$

_W_ 10. If $f(x)=e^{3\ln\left(x^2\right)}$, then $f'(x)=$

If you use one of the laws of logarithms to simplify the expression at the beginning, it will save you much time and energy: $\ln a^b=b\ln a$

$$f(x)=e^{3\ln\left(x^2\right)}=e^{\ln\left(x^2\right)^3}=e^{\ln\left(x^6\right)}=x^6$$

$$f'(x)=6x^5$$

If you didn't see that, you will still have to use the property at the end to find a matching answer.

$$f'(x)=e^{3\ln\left(x^2\right)}\bullet\frac{d}{dx}\left(3\ln\left(x^2\right)\right)$$

$$f'(x)=e^{3\ln\left(x^2\right)}\bullet\left(3\bullet\frac{1}{x^2}\bullet\frac{d}{dx}\left(x^2\right)\right)=e^{3\ln\left(x^2\right)}\bullet\frac{3\bullet 2x}{x^2}$$

$$f'(x)=e^{3\ln\left(x^2\right)}\bullet\left(\frac{6}{x}\right)=e^{\ln\left(x^2\right)^3}\bullet\left(\frac{6}{x}\right)=e^{\ln\left(x^6\right)}\bullet\left(\frac{6}{x}\right)=x^6\bullet\left(\frac{6}{x}\right)=6x^5$$

_G_ 11. $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}} =$

$$\int_0^{\sqrt{3}} \frac{dx}{\sqrt{4-x^2}} = \int_0^{\sqrt{3}} \frac{dx}{\sqrt{4\left(1-\left(\frac{x}{2}\right)^2\right)}} = \int_0^{\sqrt{3}} \frac{dx}{2\sqrt{\left(1-\left(\frac{x}{2}\right)^2\right)}} =$$

$$= Sin^{-1}\left(\frac{x}{2}\right)\Big|_0^{\sqrt{3}} = Sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - Sin^{-1}\left(\frac{0}{2}\right) = \frac{\pi}{3}$$

_B_ 12. If $\frac{dy}{dx} = 2y^2$ and if $y = -1$ when $x = 1$, then, when $x = 2,\ y =$

$$\frac{dy}{dx} = 2y^2$$
$$\int y^{-2}\,dy = \int 2dx$$
$$-y^{-1} = 2x + K$$
$$1 = 2 + K \Rightarrow \quad K = -1$$
$$-\frac{1}{y} = 2x - 1$$
$$-\frac{1}{y} = 3$$
$$y = -\frac{1}{3}$$

_E_ 13. The top of a 25-foot ladder is sliding down a vertical wall at a constant rate of 3 feet per minute. When the top of the ladder is 7 feet from the ground, what is the rate of the distance between the bottom of the ladder and the wall (in feet per minute)?

Draw a diagram (right triangle where the ladder is the hypotenuse).
Call the vertical position *y* and the horizontal position *x*.
When y = 7 ft, then x = 24 ft using the Pythagorean theorem.
This is a related rates problem, so take the derivative of both sides of the equation with respect to *t*.

$$x^2 + y^2 = 25^2 \quad \text{and} \quad \frac{dy}{dt} = -3\,\text{ft/min}$$

$$\frac{d}{dt}\left(x^2 + y^2\right) = \frac{d}{dt}\left(25^2\right)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2(24)\frac{dx}{dt} + 2(7)(-3) = 0$$

$$48\frac{dx}{dt} - 42 = 0$$

$$\frac{dx}{dt} = \frac{42}{48} = \frac{7}{8}\,\text{ft/min}$$

_R_ 14. If the graph of $y = \frac{ax+b}{x+c}$ has a horizontal asymptote $y = 2$ and a vertical asymptote of $x = -3$, then $a + c =$

$$y = \frac{ax+b}{x+c}$$

$x + c = 0$ when $x = -3$ so $c = 3$

Now solve for x:

$$xy + cy = ax + b$$

$$xy - ax = cy + b$$

$$x = \frac{cy+b}{y-a}$$

$y - a = 0$ when $y = 2$ so $a = 2$

_F_ 15. The radius of a circle is increasing at a nonzero rate, and at a certain instant, the rate of increase in the area of the circle is numerically equal to the rate of increase in its circumference. At this instant, the radius of the circle is

$C = 2\pi r \qquad A = \pi r^2$

$\frac{dC}{dt} = 2\pi\frac{dr}{dt} \qquad \frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

$2\pi\frac{dr}{dt} = 2\pi r\frac{dr}{dt} \qquad$ so r = 1

_O_ 16. The fundamental period of $2\cos(3x)$ is

3 cycles in 2π so it makes 1 cycle in $\frac{2\pi}{3}$

The droodle used in this puzzle was drawn by Roger Price and appeared in his book called *Droodles.*

Answers: (units have been omitted)

A. $-\frac{4}{3}$

B. $-\frac{1}{3}$

C. 0

D. $\frac{1}{4}$

E. $\frac{7}{8}$

F. 1

G. $\frac{\pi}{3}$

H. 1.32

I. $\frac{4}{3}$

J. 2

K. 2.55

L. nonexistent

M. $2\sqrt{3}\pi$

N. 3

O. $\frac{2\pi}{3}$

P. π

Q. 2^{x-1}

R. 5

S. $\frac{9\pi}{2}$

T. $6(\ln x)e^{3\ln(x^2)}$

U. $2^x \ln 2$

V. $\ln(x^2-2x+2)$

W. $6x^5$

X. $\frac{3}{x^2}e^{3\ln(x^2)}$

Y. $\text{Arctan}(x-1)$

Z. None of the above