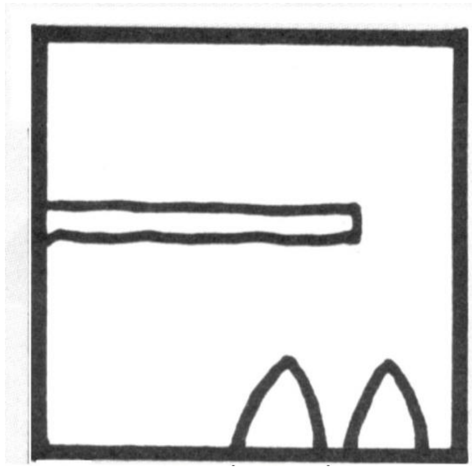
ANSWER KEY to Turvy with Integration -- by David Pleacher



Here is the title right-side-up:

"T W O   P O P E S   A T   A   G A R A G E   S A L E."

15 9 19   20 19 20 18 14   17 15   17   5 17 11 17 5 18   14 17 8 18

Here is the title upside-down:

"O L Y M P I C   D I V E: A   P E R F E C T   T E N

19 8 4 6 20 16 12   7 16 3 18   17   20 18 11 2 18 12 15   15 18 10

F O R   T O E   P O I N T S."

2 19 11   15 19 18   20 19 16 10 15 14

ANSWERS:

1. K	6. M	11. R	16. I
2. F	7. D	12. C	17. A
3. V	8. L	13. B	18. E
4. Y	9. W	14. S	19. O
5. G	10. N	15. T	20. P

Integral Problems: Answers:

1. $\int x \ln x \, dx =$ K. $\frac{x^2}{4}(2\ln x - 1) + C$

Use integration by parts. Letting u = ln(x) and dv = x dx yields

$$\int x \ln x \, dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2x} dx$$
$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$
$$= \frac{x^2}{2}(2\ln x - 1) + C$$

2. What is the area under the curve described by the parametric equations $x = \sin t$ and $y = \cos^2 t$ for $0 \le t \le \frac{\pi}{2}$? F. $\frac{2}{3}$

Convert these parametric equations into the following Cartesian equation:

$$y = 1 - x^2$$

So, the area under the curve would be given by:

$$A = \int_0^1 (1 - x^2) dx = \frac{2}{3}$$

3. $\int_1^e \left(\frac{x^2+4}{x}\right)dx=$ V. $\frac{e^2+7}{2}$

$$\int_1^e \left(\frac{x^2+4}{x}\right)dx=\int_1^e \left(x+\frac{4}{x}\right)dx$$
$$=\left(\frac{x^2}{2}+4\ln x\right)\Big|_1^e$$
$$=\left(\frac{e^2}{2}+4\ln e\right)-\left(\frac{1^2}{2}+4\ln 1\right)$$
$$=\left(\frac{e^2}{2}+4\right)-\left(\frac{1}{2}+4(0)\right)$$
$$=\left(\frac{e^2}{2}+4\right)-\frac{1}{2}=\frac{e^2+7}{2}$$

4. $\int 6x^3e^{3x}dx=$ Y. $\frac{2}{9}e^{3x}\left(9x^3-9x^2+6x-2\right)+C$

This is a very involved integration by parts. Use a chart:

u	dv	+/-1
$6x^3$	e^{3x}	+1
$18x^2$	$\frac{e^{3x}}{3}$	-1
36x	$\frac{e^{3x}}{9}$	+1
36	$\frac{e^{3x}}{27}$	-1
0	$\frac{e^{3x}}{81}$	+1

-1

$$\int 6x^3 e^{3x}dx = 2x^3 e^{3x} - 2x^2 e^{3x} + \frac{4xe^{3x}}{3} - \frac{4}{9}e^{3x} + C$$

$$= \frac{2}{9}e^{3x}\left(9x^3 - 9x^2 + 6x - 2\right) + C$$

5. $\int_0^1 e^{2x}\,dx =$ G. $\frac{e^2 - 1}{2}$

Solve by u-substitution. Let u = 2x, so du = 2dx.

$$\int_0^1 e^{2x}\,dx = \frac{1}{2}\int_0^2 e^u\,du = \frac{e^2 - 1}{2}$$

6. If $F(x) = \int_2^{x^2} t^2\,dt$, Then $F(2) =$ M. $\frac{56}{3}$

$$F(2) = \int_2^4 t^2\,dt = \frac{56}{3}$$

7. Which of the following are antiderivatives of $f(x)=\cos^3 x\sin x$?

I. $F(x)=\dfrac{-\cos^4 x}{4}$

II. $F(x)=\dfrac{\sin^2 x}{2}-\dfrac{\sin^4 x}{4}$

III. $F(x)=\dfrac{1-\cos^4 x}{4}$

D. I, II, and III

Here, we should take the derivative of each I, II, and III and see what we get.

$$\frac{d}{dx}\left(\frac{-\cos^4 x}{4}\right)=\frac{-4\cos^3 x}{4}(-\sin x)=\cos^3 x\sin x$$

$$\frac{d}{dx}\left(\frac{\sin^2 x}{2}-\frac{\sin^4 x}{4}\right)=\sin x\cos x-\frac{4\sin^3 x\cos x}{4}=\sin x\cos x-\sin^3 x\cos x$$

$$=\sin x\cos x\left(1-\sin^2 x\right)=\sin x\cos x\left(\cos^2 x\right)=\cos^3 x\sin x$$

$$\frac{d}{dx}\left(\frac{1-\cos^4 x}{4}\right)=\frac{-4\cos^3 x\,(-\sin x)}{4}=\cos^3 x\sin x$$

8. $\displaystyle\int_0^{\frac{\pi}{4}}\sin 2x\,dx=$

L. $\dfrac{1}{2}$

This is a simple u-substitution integral.
Let u = 2x, so du = 2dx.

It follows that $\displaystyle\int_0^{\pi/4}\sin 2x\,dx$ becomes

$$\frac{1}{2}\int_0^{\pi/2}\sin u\,du=-\frac{1}{2}\cos u\Big|_0^{\pi/2}$$

$$=-\frac{1}{2}(0-1)=\frac{1}{2}$$

9. $\int_0^{\pi/3} \left(\frac{\tan x\, e^{\sec x}}{\cos x} \right) dx =$ W. $e^2 - e$

We must recognize that $\frac{1}{\cos x} = \sec x$. This lets us rewrite the integral as

$$\int_0^{\pi/3} \left(\sec x \tan x e^{\sec x} \right) dx$$

Next we can evaluate the integral using u-substitution.
If we let u = sec(x) and du = sec(x) tan(x) dx, we get

$$\int_1^2 e^u\, du = e^u \Big|_1^2 = e^2 - e$$

10. $\int_0^1 \left(\sqrt{x} \right) \left(x^2 + 3x - 8 \right) dx =$ N. $\frac{-404}{105}$

Before we try to do anything, distribute the $\sqrt{x}$ and change the notation to that of rational exponents. After these two steps, we get

$$\int_0^1 \left(x^{\frac{5}{2}} + 3x^{\frac{3}{2}} - 8x^{\frac{1}{2}} \right) dx$$

Integrating leads to

$$\left(\frac{2}{7} x^{\frac{7}{2}} + \frac{6}{5} x^{\frac{5}{2}} - \frac{16}{3} x^{\frac{3}{2}} \right) \Bigg|_0^1 = \left(\frac{2}{7} + \frac{6}{5} - \frac{16}{3} \right) - 0 = \frac{-404}{105}$$

11. $\int_1^5 \left(\frac{3x}{x^3} \right) dx =$ R. $\frac{12}{5}$

$$\int_1^5 \left(\frac{3x}{x^3} \right) dx = \int_1^5 \left(\frac{3}{x^2} \right) dx = \frac{-3}{x} \Big|_1^5 = \frac{-3}{5} + 3 = \frac{12}{5}$$

12. What are all the values of k such that $\int_{-2}^{k} x^3\,dx = 0$? C. − 2 and 2

Evaluate the definite integral and apply the fundamental Theorem:

$$\int_{-2}^{k} x^3\,dx = \left.\frac{x^4}{4}\right|_{-2}^{k} = 0$$

$$\frac{k^4}{4} - \frac{2^4}{4} = 0$$

$$k^4 - 16 = 0$$

$$k = \pm 2$$

13. $\int_{0}^{\pi/4} \sin x \cos x\,dx =$ B. $\frac{1}{4}$

This is a straight-forward u substitution integration problem.
If we let u = sin x, then du = cos x dx and

$$\int_{0}^{\pi/4} \sin x \cos x\,dx = \int_{0}^{\sqrt{2}/2} u\,du = \left.\frac{u^2}{2}\right|_{0}^{\sqrt{2}/2} = \frac{1}{4}$$

14. $\int x \sec^2 x\,dx =$ S. $x \tan x + \ln|\cos x| + C$

Use integration by parts. Let u = x and dv = $\sec^2$x dx.

$$\int x \sec^2 x\,dx = x \tan x - \int \tan x\,dx$$

$$= x \tan x + \ln|\cos x| + C$$

15. $\int_1^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx =$ T. $2e\left(e^2-1\right)$

This is a rather complicated u-substitution integration problem.

If we let $u=\sqrt{x},\ \text{ then } du=\frac{dx}{2\sqrt{x}}.$

$$\int_1^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = 2\int_1^3 e^u\ du$$

$$= 2e^3-2e = 2e\left(e^2-1\right)$$

16. $\int e^x \sin x\, dx =$ I. $\frac{1}{2}e^x\left(\sin x-\cos x\right)+C$

This is an integration by parts with a twist at the end.

Let u = sin x and dv = e^x dx

So, $\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx.$

We need to integrate by parts again. Let u = cos x and dv = e^x dx.

$$\int e^x \sin x dx = e^x \sin x - e^x \cos x - \int e^x \sin x dx + C.$$

Here's the twist. We are going to add $\int e^x \sin x dx$ to both sides of the equation.

$$2\int e^x \sin x dx = e^x \sin x - e^x \cos x + C$$

To solve for $\int e^x \sin x dx$, we will divide both sides by 2:

$$\int e^x \sin x dx = \frac{1}{2}\left(e^x \sin x - e^x \cos x\right)+C$$

17. $\int \frac{4\,dx}{\sqrt{64-16x^2}} =$ A. $\arcsin\frac{x}{2}+C$

This is definitely an arcsin problem, but it's much easier if you factor out a 16 from the denominator and simplify first.

$$\int \frac{4\,dx}{\sqrt{64-16x^2}} = \int \frac{4\,dx}{\sqrt{16}\sqrt{4-x^2}} = \int \frac{dx}{\sqrt{4-x^2}} = \arcsin\frac{x}{2}+C$$

18. $\displaystyle\int \frac{\tan x\, dx}{\sin^2 x\sqrt{\cot^2 x - 16}} =$ E. $\displaystyle\frac{-1}{4}\text{arcsec}\left(\frac{|\cot x|}{4}\right) + C$

Your instinct should tell you that this an arcsec problem, since there is a radical in the denominator and the order of subtraction is *variable – constant*. However, if u = cot x, shouldn't there be a cot x in front of the radical to match the correct denominator form of $x\sqrt{x^2 - u^2}$? Watch what happens when you rewrite the trig functions using the reciprocal identities:

$$\int \frac{\tan x\, dx}{\sin^2 x\sqrt{\cot^2 x - 16}} = \int \frac{\csc^2 x\, dx}{\cot x\sqrt{\cot^2 x - 16}}$$

Now, if u = cot x, then du = $-\csc^2 x$ dx, so – du = $\csc^2 x$ dx. Also, a = 4. So,

$$\int \frac{\csc^2 x\, dx}{\cot x\sqrt{\cot^2 x - 16}} = \frac{-1}{4} arc\sec\frac{|\cot x|}{4} + C$$

19. $\displaystyle\int \frac{e^{\tan x}}{1 - \sin^2 x} dx =$ O. $e^{\tan x} + C$

Use trig identities to write: $\displaystyle\int \frac{e^{\tan x}}{1 - \sin^2 x} dx = \int \frac{e^{\tan x}}{\cos^2 x} dx = \int e^{\tan x} \sec^2 x\, dx$

Note that this is just an e^u du problem where u = tan x and du = $\sec^2 x$ dx.

20. $\left\{\begin{array}{l} \text{If } \displaystyle\int_0^3 f(x)dx = 10 \quad \text{and} \quad \int_3^0 g(x)dx = 12 \\ \text{Then evaluate } \displaystyle\int_0^3 \left(g(x) - 3f(x)\right)dx \end{array}\right.$ P. -42

First of all, get the boundaries to match up.

According to definite integral properties, If $\displaystyle\int_3^0 g(x)dx = 12$, Then $\displaystyle\int_0^3 g(x)dx = -12$

$$\int_0^3 \left(g(x) - 3f(x)\right)dx = \int_0^3 g(x)dx \; - 3\int_0^3 f(x)dx = -12 - 3 \bullet 10 = -42$$